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Cross-waves. Part 1. Theory

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An instability mechanism, leading to the generation of cross-waves in a closed channel, was examined recently by Garrett (1970). His theory is not applicable to long channels where the wavemaker produces a primary field which is a progressive wave train. In such cases, the heaving of the mean surface, of considerable significance in the instability mechanism, is confined to the non-propagating field near the wavemaker. Here the theory of resonant interactions is extended to describe the energy transfer from this forced localized field to the cross-wave field. There are close analogies between the present results and Garrett's, although the resonant bandwidth estimated here is an order of magnitude smaller. The theory indicates that nonlinear effects may control the decay of cross-waves down the channel.

1. Introduction

When a wavemaker operates at one end of a channel, whose width is rather larger than the wavelength of the primary waves being generated, waves may appear with their crests at right angles to the wavemaker. These waves, whose frequency is generally half that of the wavemaker, are known as cross-waves. A brief historical account may be found in a recent paper by Garrett (1970), who developed a theory for the generation of cross-waves in a tank with a rigid wall opposite the wavemaker. It is an intriguing feature of Garrett's analysis that the only features of the basic flow which contribute to the instability are the mean surface displacement and the mean of the second derivative of the velocity potential at the surface. Moreover, both these quantities can be related to the performance of the wavemaker by global conservation considerations for the basic flow. Once these identifications have been made the basic flow plays no further role in the theory. It is irrelevant whether the basic flow has standing waves or is merely a non-periodic flow with waves travelling back and forth along the channel. The global identifications, however, assume the dominance of the basic flow and so, by its very nature, the theory cannot be expected to provide answers about the manner in which the basic flow is modified by the occurrence of cross-waves. Presumably such answers can only be found in the study of the detailed dynamics of the fluid, a dauntingly difficult task.

Garrett shows that the energy of the cross-wave field can be supplied by the

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work done by the wavemaker against the depth-independent second-order pressure under a standing wave. This would still apply in an open channel, as Garrett observes, but a satisfactory explanation of the instability requires the demonstration that the phase relationships are appropriate to enable the above energy transfer to occur. Because the basic details of the fluid motion are irrelevant in Garrett's calculations it might be thought that the phase relationships for the open channel could be obtained from his analysis. There are two difficulties associated with this suggestion, and these are partially concealed in the mathematical forms he obtains in terms of variables rendered non-dimensional with respect to the length of the channel. The first difficulty is that the two mean quantities tend to zero, for a given wavemaker motion, as the channel length tends to infinity. This implies that the resonant bandwidth and the growth rate tend to zero, and the instability disappears. If one invokes a dissipative mechanism to limit the channel length, as Garrett does in a different context, the decay rate due to dissipation should also be included and whether or not there is an instability then becomes uncertain. Second, even for long channels with negligible dissipation, Garrett's analysis becomes inappropriate since it is based on an independent discrete-mode analysis. For long enough channels, the lower harmonics of the Fourier decomposition along the channel will still be long waves, and they too are likely to be involved in the instability. It is the aim of this paper to extend Garrett's analysis to overcome these two difficulties for the case where the channel length is effectively infinite.

For a monochromatic progressive wave train in the absence of a wavemaker, the author has been unable to find a 'resonant interaction' which transfers energy from a progressive wave to a standing cross-wave. Any such interaction must involve a receptor for the momentum flux of the progressive wave train, which cannot be accepted by either of the travelling modes contributing to the cross-wave field. Thus a four-mode interaction at least would be involved, and if the half-frequency result is used as a guide far too many restrictions appear. Even if such an interaction does exist theoretically, it must be of such high order that its practical occurrence is rendered most unlikely by dissipative effects. It may be relevant to note that cross-waves do not appear to have been observed in naturally generated wave systems. Altogether this suggests that, to account for the origin of cross-waves, one should be more concerned with the field close to the wavemaker than with the progressive wave train. This near field admits no simple analytic description, but its Fourier transform can be obtained. As Garrett's theory is equivalent to this transform containing a δ-function term at zero wavenumber, it is natural to ask whether a continuous spectrum over a band in which the wavenumbers are small can drive a similar instability. It will be shown that this can happen.

Instability calculations, based on resonant-interaction theory, normally involve discrete-mode analyses. Thus the disturbance is taken as a monochromatic wave and its growth rate calculated. However, in a case where only a narrow band of the spectrum of the driving term is expected to be involved, one must anticipate that a discrete-mode representation of the disturbance is also inappropriate, and some modification of the usual calculational procedures will be necessary. A pattern of calculation is developed at first for a simple model equation where the algebraic difficulties do not obscure the essentials. This serves as a guide as to how to carry out the full calculations in such a way as to render the algebra possible.

The basic results obtained are very similar in form to those obtained by Garrett. The main differences lie in the scales on which the effects occur. Thus if ϵ is the small parameter measuring the amplitude of the waves generated, then the bandwidth of frequencies for which cross-waves will be observed is reduced from $O(\epsilon)$ to $O(\epsilon^2)$, and the growth rate is correspondingly reduced. This implies that, for a given level of dissipation, larger motions will be required in order to generate cross-waves in a longer channel. However, during the initial stages, where the theory applies, there is a variation in the amplitude of the cross-waves along the channel. Garrett's result that the length scale of such variations is infinite is modified so that the length scale becomes of $O(\epsilon^{-1})$, provided that this is not larger than the scale produced by viscous decay. There is also a close analogy with Garrett's condition for the reduction in the generation of cross-waves but his heuristically introduced length parameter does not appear. The theory does not clarify the questions concerning the manner in which the basic flow is modified by the appearance of cross-waves. Transfer of energy from the basic flow field will affect the phase of the pressure on the wavemaker, and hence may be expected to modify the energy and momentum fed into the travelling-wave mode but any analytic calculation of such effects appears improbable.

2. Basic equations

Consider a wavemaker operating at one end of a uniform horizontal channel of breadth b in which the undisturbed depth of liquid is d. If the angular frequency of the periodic motion of the wavemaker is ω and g is the acceleration due to gravity, take ω^{-1} as the unit of time, $g\omega^{-2}$ as the unit of length and $g\omega^{-1}$ as the unit of velocity. Take rectangular Cartesian axes $Oxyz$ with the x axis along the channel, with origin related to the central position of the wavemaker, and the z axis vertically upward with its origin at the undisturbed liquid surface. In these co-ordinates, and with the above choice of units, let $\phi(x,y,z,t)$ be the velocity potential, $Z(x,y,t)$ the vertical co-ordinate of the surface and $\epsilon F(z)\sin t$ the x co-ordinate of the wavemaker. The flow is assumed to be inviscid and irrotational, so that the basic equation governing the flow is

$$\nabla^2\phi = 0,$$

to be satisfied everywhere in the fluid. The associated boundary conditions are

$$\phi_z(x,y,-\beta) = 0,$$

$$\phi_x - \epsilon F'(z)\,\phi_z \sin t = \epsilon F(z)\cos t \quad \text{on} \quad x = \epsilon F(z)\sin t,$$

$$\phi_y(x,0,z) = 0, \quad \phi_y(x,\pi\alpha^{-1},z) = 0,$$

and on the free surface $z = Z(x, y, t)$

$$\phi_z = Z_t + \phi_x Z_x + \phi_y Z_y,$$

$$\phi_t + Z + \tfrac{1}{2}\{\phi_x^2 + \phi_y^2 + \phi_z^2\} = 0,$$

together with a radiation condition at $x = \infty$. In the above

$$\alpha = \pi g/(b\omega^2), \quad \beta = d\omega^2/g.$$

This system has as its solution a velocity potential ϕ and a surface displacement η, both of which are independent of y. Consider now an infinitesimal disturbance of the basic flow in the lowest cross-wave mode. Results for higher cross-wave modes may be inferred directly from the subsequent analysis by replacing α by $n\alpha$ everywhere except in the definition in terms of physical quantities. It is assumed that second-order quantities in the cross-wave amplitude are negligible, so that the resulting theory will be restricted to the discussion of the onset of the instability. The disturbed velocity potential takes the form

$$\phi(x, z, t) + \delta\Phi(x, z, t) \cos \alpha y,$$

and the equation to the free surface the form

$$z = \eta(x, t) + \delta\zeta(x, t) \cos \alpha y.$$

The equation satisfied by the cross-wave field is

$$\Phi_{xx} + \Phi_{zz} - \alpha^2 \Phi = 0$$

and the boundary conditions are

$$\Phi_z = 0 \quad \text{on} \quad z = -\beta,$$

$$\Phi_x = \epsilon F'(z)\, \Phi_z \sin t \quad \text{on the wavemaker}$$

and the free-surface conditions

$$\Phi_z + \phi_{zz}\zeta = \zeta_t + \phi_x \zeta_x + \Phi_x \eta_x, \tag{1a}$$

$$\Phi_t + \phi_{tz}\zeta + \phi_x \Phi_x + \phi_z \Phi_z + \zeta = 0 \tag{1b}$$

on the surface $z = \eta(x, t)$ of the basic flow.

So far no use has been made of the smallness of ϵ, a measure of the wave amplitude, and the boundary conditions (1) may be simplified by replacing ϕ and η by their approximations in powers of ϵ. For much of the analysis terms which are of $O(\epsilon^2)$ may be ignored, and such approximations will now be made. Later it will become necessary to return to equations (1) to check on the relative importance of the terms neglected. The appropriate first-order approximations to equations (1) are

$$\Phi_z + \Phi_{zz}\eta + \phi_{zz}\zeta = \zeta_t + \phi_x \zeta_x + \Phi_x \eta_x, \tag{2a}$$

$$\Phi_t + \Phi_{tz}\eta + \phi_{tz}\zeta + \phi_x \Phi_x + \phi_z \Phi_z + \zeta = 0, \tag{2b}$$

where all derivatives of ϕ and Φ are to be evaluated on $z = 0$. In these equations ϕ and Φ should be interpreted as the first-order description of these quantities rather than the exact descriptions. The first-order field consists of a uniform wave train, generated at the wavemaker and travelling down the channel, together with a forced, periodic, non-propagating field near the wavemaker. For the remainder of this section it will be assumed that the instability mechanism producing cross-waves is associated with the pulsing flow near the wavemaker, for reasons which have been discussed in the introduction. Actually this is not quite true, but is sufficiently close to the truth to form a basis for a preliminary investigation. Thus the periodic-wave-train part of the basic flow solution will be ignored, and it will be left to an *a posteriori* verification to show that this is a consistent approximation procedure. Such a method of investigation may cause one to miss certain forms of instability but is a valid means of investigating a given style of mechanism. In the present context the assumption appears to be the natural generalization of Garrett's organization of his calculations, which was based on the presumption that it is essentially the heaving of the mean surface, rather than the standing waves, which feeds the cross-waves. For the closed channel the agreement with experiment is most satisfactory.

Thus in equations (2) ϕ and η may be considered as being quite complicated functions of x times $\cos t$. However, there is a not too unreasonable expression for the Fourier transforms of these functions of x. A conceptual scheme for solving equations (2) is to undertake a perturbation expansion in powers of ϵ, and if such a procedure were followed and the equations for the first two orders were combined, there would result, after the elimination of η, an equation of the form

$$\Phi_{tt} + \Phi_z = \epsilon \sum_i a_i L_i(\Phi)\, M_i(\phi) + O(\epsilon^2), \tag{3}$$

where the a_i are constants and L_i and M_i are linear operators. The fact that ϕ is really only accessible through its Fourier transform, and, moreover, the fact that Fourier methods are the most tractable means of handling the presence of the z derivative which occurs, indicates that one should work with the Fourier transform of (3). This leads one to consider an equation of the form

$$\hat{\Phi}_{tt} + \{\Omega(k)\}^2 \hat{\Phi} = (2\pi)^{-1} \epsilon \sum_i a_i \int_{-\infty}^{\infty} \hat{L}_i\{\hat{\Phi}(k,t)\}\, \hat{M}_i\{\phi(k-k') \cos t\}\, dk',$$

where $\Omega(k)$ is the angular frequency of a free cross-wave with wavenumber component k in the direction of the channel and ^ denotes Fourier transforms. The calculational procedure to be adopted is rather long and involved, and raises questions as to the validity of using this approximate equation. Moreover, the discussion of such a differential–integral equation is somewhat novel. Thus the methods to be adopted are considered in the first instance for a simple model equation suggested by the above. The experience gained with this simpler equation will serve as a valuable guide for dealing with the more complicated system.

3. Model calculation

Consider the differential–integral equation

$$u_{tt} + \Omega^2 u = \epsilon \cos t \int_{-\infty}^{\infty} F(k,k')\, u(t,k')\, dk', \tag{4}$$

where F is a given smooth function of its arguments, and it is desired to investigate whether, for large times, the behaviour of solutions is significantly affected by the small term on the right-hand side. First, a simple perturbation expansion

$$u = \sum_{i=0}^{\infty} \epsilon^i u_i(t, k)$$

is tried, and a representative first approximation is

$$u_0 = \alpha(k) \cos \Omega(k)\, t.$$

The corresponding second approximation satisfies

$$u_{1tt} + \Omega^2 u_1 = \cos t \int_{-\infty}^{\infty} F(k, k')\, \alpha(k') \cos \Omega' t\, dk',$$

where Ω' denotes $\Omega(k')$. A particular integral for u_1 is

$$\Omega^{-1} \int_0^t \sin \Omega(t-s) \cos s \int_{-\infty}^{\infty} F(k, k')\, \alpha(k') \cos \Omega' s\, dk'\, ds.$$

One may argue heuristically that the formal perturbation expansion should give a useful description of solutions provided that ϵu_1 remains small in comparison with u_0. It may be seen that this may not always be so for large values of t. It seems reasonable to assume that the contribution from small or moderate values of s does not invalidate the expansion. Thus, in any examination for growth of the particular integral for large time, it suffices to expand the inner integral for large values of s. If the initial disturbance is not highly structured the initial spectrum $\alpha(k')$ will also be a smooth function, and then the inner integral will be entirely dominated by the contributions from the neighbourhoods of any points of stationary phase. For the dispersion relation

$$\Omega(k') = \{(\alpha^2 + k'^2)^{\frac{1}{2}} \tanh |(\alpha^2 + k'^2)^{\frac{1}{2}} \beta|\}^{\frac{1}{2}} \tag{5}$$

for cross-waves on water, there is only one point of stationary phase, corresponding to $k' = 0$. Then if one writes

$$\Omega_0 = \Omega(0) = \{\alpha \tanh \alpha\beta\}^{\frac{1}{2}}, \tag{6}$$

the only way that u_1 may be so large as to invalidate, automatically, the perturbation scheme is for

$$\int_0^t s^{-\frac{1}{2}} \sin \Omega(k)\, (t-s) \cos s \cos \Omega_0 s\, ds$$

to become large with t. In general the integrand is oscillatory, and then the Riemann–Lebesque lemma guarantees that the integral is not large. However, there are certain values of k at which the integrand ceases to be an oscillatory function, and then the integral is of $O(t^{\frac{1}{2}})$ for large values of t. An elementary argument using the trigonometric addition formulae shows that the critical values of k are such that one of the combinations $t\{\Omega \pm 1 \pm \Omega_0\}$ is not large when t is large. There are a number of values k_0, zeros of one of the four combinations $\{\Omega \pm 1 \pm \Omega_0\}$, such that in a narrow band of wavenumbers about them the first correction ϵu_1

will grow in time to exceed u_0 in order of magnitude. A typical such k_0, satisfying

$$\{(\alpha^2+k_0^2)^{\frac{1}{2}}\tanh|(\alpha^2+k_0^2)^{\frac{1}{2}}\beta|)^{\frac{1}{2}} = 1+|\alpha\tanh\alpha\beta|^{\frac{1}{2}},$$

exists for all values of α and β. As the group velocity does not vanish at such a k_0, the width of the band of wavenumbers around k_0 in which the contribution to u_1 will grow with time will be of $O(t^{-1})$. Of course, this estimate, based on a regular perturbation procedure, ceases to be justified once ϵu_1 is comparable with u_0, so that it is no longer valid when $\epsilon t^{\frac{1}{2}}$ is not small. The ideas underlying the method of matched asymptotic expansions (e.g. Cole 1968) suggest that, on a time scale of $O(\epsilon^{-2})$ in a wavenumber band of width $O(\epsilon^2)$, u_0 may cease to be a very good approximation to u. However, the present differential–integral equation displays somewhat unusual properties in comparison with differential equations where these ideas have been so effective. For, a modification to u_0 in such a narrow bandwidth will produce an $O(\epsilon^3)$ change on the right-hand side of (4), and this cannot be expected to modify the determination of u_0 on a time scale of $O(\epsilon^{-2})$. Thus there seems reason to believe that when ϵu_1 becomes comparable with u_0 the estimate for u_1 remains appropriate. If this is so, the bandwidth will continue to shrink as t^{-1}, while ϵu_1 will grow as $\epsilon t^{\frac{1}{2}}$ and its contribution to the right-hand side of (4) will decay as $\epsilon t^{-\frac{1}{2}}$. Such a contribution would not seem to disturb the usefulness of a perturbation scheme. Furthermore, in complete contrast to the usual pattern which occurs in discrete model analysis, there is no reinforcement of the growth since the growing term is proportional to $\alpha(0)$ rather than $\alpha(k_0)$, nor is there any growth at $k=0$ due to $\alpha(k_0)$. How one could compute the large-time behaviour of $\alpha(k)$ near $k=k_0$ is not clear to the author. There may be a higher order closure which permits the limitation or modification of this growth, or the growth indicated by u_1 may continue until limited by some dissipative mechanism. But whichever is the case is not important, for the earlier argument shows that u_0 will not be affected on a time scale of $O(\epsilon^2)$. Furthermore, it must be remembered that u is a Fourier transform of a physical variable, so that even unit order changes of u over a wavenumber band of $O(\epsilon^2)$ will result in very small changes in the corresponding physical variable. It therefore appears justified to believe that such cases do not indicate a real instability with a growth rate of $O(\epsilon^2)$.

An examination of the various possibilities reveals that there is only one way in which the resonance conditions may be satisfied and also not be ruled out on the above grounds. This happens when the combination

$$\{(\alpha^2+k^2)^{\frac{1}{2}}\tanh|(\alpha^2+k^2)^{\frac{1}{2}}\beta|\}^{\frac{1}{2}} = 1-|\alpha\tanh\alpha\beta|^{\frac{1}{2}}$$

leads to the root $k=0$. Then the pattern is different in two significant ways. First, the group velocity vanishes, so that the Riemann–Lebesque lemma cannot be invoked for k^2t not large, and the bandwidth affected is increased to $O(t^{-\frac{1}{2}})$. Thus the contribution of ϵu_1 to the integral in (4) will not decay in time. Moreover, it will occur in that part of the range of integration which contributes a growing term to u_1, and hence it is continually reinforcing its own changes. Thus it appears that this case provides the opportunity for a parametric resonance of the type which gave Garrett his instability for the case when ϕ had a discrete spectrum. In particular, one may note that this combination, with $k = 0$, leads to the condition $\Omega_0 = \frac{1}{2}$, which is the observed frequency for cross-waves. It also corresponds to Garrett's analysis, in which it is the x-independent mode of the basic flow which drives the instability. However, the fact that there is a wave band involved indicates that there would be a slow x variation in the cross-wave field.

The close analogy with Garrett's analysis is comforting, and suggests that a further examination of the long-wave band is justified. The simple perturbation analysis is not valid once $\epsilon t^{\frac{1}{2}}$ is not small, and the usual heuristic arguments suggest looking for a solution in the small-wavenumber band in the form $U(t, \tau, \kappa)$, where $\tau = \epsilon^2 t$ and $\kappa = \epsilon^{-1} k$. There are certain subtleties about the method of approximation which call for comment. If an instability were to occur with an exponential growth rate of $O(\epsilon^2)$ in the long-wave band, it would not be true that, for wavenumbers outside the scaled range, the changes in the value of $u(k, t)$ are necessarily unimportant. Equation (4) implies that an exponential growth in any wavenumber range is reflected throughout the whole spectrum. A bandwidth of $O(\epsilon)$ for resonance implies, however, that it will produce a change in u with the same exponential growth rate, but will have a small factor of $O(\epsilon^2)$ for k other than small. The inclusion of such a small exponential term in the other parts of the spectrum of u would affect only the ϵ^3 term in the discussion of the determination of u in the small-wavenumber band. It thus follows that the growth, if any, of $U(t, \tau, \kappa)$ on a time scale of $O(\epsilon^{-2})$ may be discussed independently of the behaviour of u in other parts of the spectrum.

Parametric resonance phenomena will be excited not only when the condition $\Omega_0 = \frac{1}{2}$ is satisfied exactly, but also for values of the parameters which lead to this condition being satisfied closely enough. Experience suggests that Ω_0 can differ from $\frac{1}{2}$ by a quantity of the same order as the growth rate which may be involved, and parametric resonance still occur. Thus one introduces a parameter λ, defined by

$$\alpha \tanh \alpha\beta = \Omega_0^2 = \tfrac{1}{4} + \lambda\epsilon^2,$$

and seeks a range of values of λ, expected to be of unit order, for which U will grow in time. A wrong guess for this order could be corrected as it became apparent in subsequent analysis, but in fact the guess proves to be correct. Then for the long-wave band a suitable approximation for Ω^2 is given by

$$(k^2 + \alpha^2)^{\frac{1}{2}} \tanh |(k^2 + \alpha^2)^{\frac{1}{2}} \beta| = \tfrac{1}{4} + \lambda\epsilon^2 + \gamma\epsilon^2\kappa^2 + O(\epsilon^2),$$

and the parameter γ, so defined, is positive since $z \tanh z$ is a strictly increasing function of z for positive values of z. Thus (4) may be arranged in the form

$$U_{tt} + (\tfrac{1}{4} + \lambda\epsilon^2 + \gamma\epsilon^2\kappa^2)\, U + 2\epsilon^2 U_{t\tau} = \epsilon^2 F_0 \int_{-\infty}^{\infty} U(t, \tau, \kappa')\, d\kappa' \cos t + R(t, \tau, k, \epsilon), \quad (7)$$

where F_0 denotes $F(0, 0)$ and R denotes terms which are expected not to contribute to changes of U of unit order on a time scale of $O(\epsilon^{-2})$ and hence may be ignored in subsequent discussion.

A standard argument based on the method of multiple scales (Cole 1968) may now be applied. Consideration of the leading approximation to (7) yields

$$U_0 = V(\tau,\kappa)\cos\tfrac{1}{2}t + W(\tau,\kappa)\sin\tfrac{1}{2}t, \tag{8}$$

and a second approximation is now sought. For general functions V and W the second approximation will not be a bounded function of the variable t, but this difficulty may be avoided by requiring that

$$V_\tau - (\lambda+\gamma\kappa^2)\, W = \tfrac{1}{2}F_0\int_{-\infty}^{\infty} W(\tau,\kappa')\,d\kappa' = X(\tau) \tag{9a}$$

and
$$W_\tau + (\lambda+\gamma\kappa^2)\, V = \tfrac{1}{2}F_0\int_{\infty}^{\infty} V(\tau,\kappa')\,d\kappa' = Y(\tau), \tag{9b}$$

where these equations define X and Y. The argument leading to these differential–integral equations is almost identical with a multi-scaling discussion of the Mathieu equation. Equations (9) may be solved by treating X and Y as known functions and solving the resultant differential equations. Thus it can be shown that

$$V(\tau,\kappa) = \int_0^\tau \{X'(s)\,\delta^{-1} + Y(s)\}\sin\delta(\tau-s)\,ds + V(0,\kappa), \tag{10a}$$

$$W(\tau,\kappa) = \int_0^\tau \{Y'(s)\,\delta^{-1} - X(s)\}\sin\delta(\tau-s)\,ds + W(0,\kappa), \tag{10b}$$

where
$$\delta = \lambda+\gamma\kappa^2.$$

If these results are now integrated with respect to κ and the definitions (9) of X and Y are used, it is easy to show that

$$X = X_0 + \tfrac{1}{2}F_0\int_0^\tau \{Y'(s)\,K_1(\tau-s) - X(s)\,K_2(\tau-s)\}\,ds \tag{11a}$$

and
$$Y = Y_0 + \tfrac{1}{2}F_0\int_0^\tau \{X'(s)\,K_1(\tau-s) + Y(s)\,K_2(\tau-s)\}\,ds, \tag{11b}$$

where the kernels are given by

$$K_1(s) = \int_{\infty}^{\infty} (\lambda+\gamma\kappa^2)^{-1}\sin(\lambda+\gamma\kappa^2)\,s\,d\kappa$$
$$= -(\pi/\gamma)^{\frac{1}{2}}\int_s^{\infty} \sigma^{-\frac{1}{2}}\cos(\lambda\sigma+\pi/4)\,d\sigma$$

and
$$K_2(s) = \int_{\infty}^{\infty} \sin(\lambda+\gamma\kappa^2)\,s\,d$$
$$= (\pi/\gamma)^{\frac{1}{2}}\,s^{-\frac{1}{2}}\sin(\lambda s+\tfrac{1}{4}\pi).$$

The second integral is to be found on p. 397 of Gradshteyn & Ryzhik (1965) and the first may be found by an appropriate integration of the cosine integral corresponding to the second. The integral equations (11) are of Volterra form, and the application of a Laplace transform is clearly indicated. If a bar is used to denote the result of a Laplace transformation and p is the transform variable, then the integral equations reduce to

$$\bar{X}\{1+\tfrac{1}{2}F_0\bar{K}_2\}-\tfrac{1}{2}F_0 p\bar{K}_1\bar{Y} = X_0/p,$$
$$-\tfrac{1}{2}F_0 p\bar{K}_1\bar{Y}+\{1-\tfrac{1}{2}F_0\bar{K}_2\}\bar{Y} = Y_0/p.$$

The transforms $\bar{K}_1$ and $\bar{K}_2$ can be obtained in terms of the known transforms (Erdélyi *et al.* 1954)

$$s^{-\frac{1}{2}}\sin s\lambda \to \tfrac{1}{2}i\Gamma(\tfrac{1}{2})\,[(p+i\lambda)^{-\frac{1}{2}}-(p-i\lambda)^{-\frac{1}{2}}],$$
$$s^{-\frac{1}{2}}\cos s\lambda \to \tfrac{1}{2}\Gamma(\tfrac{1}{2})\,[(p+i\lambda)^{-\frac{1}{2}}+(p-i\lambda)^{-\frac{1}{2}}].$$

These transforms are free of singularities in the right half of the complex-p plane and hence any solutions of the integral equations (11) which grow exponentially with τ must be associated with zeros of the transfer function which lie in the right half plane. The equations for $\bar{X}$ and $\bar{Y}$ are singular if

$$\begin{aligned} 0 &= 1-\tfrac{1}{4}F_0^2(\bar{K}_2^2+p^2\bar{K}_1^2) \\ &= 1-\tfrac{1}{4}F_0^2(\pi/\gamma)\,\{\Gamma(\tfrac{1}{2})\}^2\,(p^2+\lambda^2)^{-\frac{1}{2}}, \end{aligned}$$

which results after some manipulation. It thus follows that there is a positive real root p_0, given by

$$p_0 = \{\tfrac{1}{16}\pi^4F_0^4\gamma^{-2}-\lambda^2\}^{\frac{1}{2}},$$

if

$$|\lambda| < \tfrac{1}{4}\pi^2F_0^2\gamma^{-1}.$$

For other than very special initial conditions an exponentially growing term will appear in both X and Y. The variables V and W are determined from X and Y using (10). A little computation shows that at least one of V and W must have a similar exponentially growing factor, and consequently so must U. It may be observed that (10) imply that all wavenumbers have the same exponential growth rate, but in that part of the spectrum where $\kappa^2\tau$ is large the overall amplitude is smaller.

4. Application to water waves

The ideas which have been developed in § 3 will now be applied to the much more complicated equations governing cross-waves. The model equation (4) was chosen to be representative of the first-order approximation in ϵ to the equations for water waves. However, for the model equation, the instability occurred on a time scale of $O(\epsilon^{-2})$, and, if the same pattern of calculation is to be applicable, clearly it will be necessary to show that in the water-wave equations no second-order terms in ϕ may affect the calculations profoundly. This implies that terms in both Φ and ϕ, correct to second order, must be considered. The number of terms involved is alarmingly large, and the result is that almost all of them can be neglected. The detailed discussion of all these terms does not seem warranted, but some account of the manner in which various terms may be assessed does seem indicated. The procedure of § 3 involved a perturbation expansion in powers of ϵ, and the same method will be applied for Φ. Because Φ satisfies a linear set of equations, the crucial surface conditions will yield successively equations of the form

$$\Phi_{0tt}+\Phi_{0z} = 0,$$
$$\Phi_{1tt}+\Phi_{1z} = L_1(\Phi_0),$$
$$\Phi_{2tt}+\Phi_{2z} = L_1(\Phi_1)+L_2(\Phi_0),$$

where L_1 is a linear operator calculable from the first-order basic flow and L_2 is a linear operator derived from second-order terms (not merely the second-order terms in the basic flow). Now, any term which appears on the right-hand side in the *a* form like $g(x,t)\,\Phi_0$, where g has a smooth Fourier transform, will appear in the subsequent analysis in a convolution integral with Φ. Because only a narrow wave band will be under consideration (the same resonance conditions as in § 3 apply), this will imply that such terms should be considered one stage later in the calculation. Thus terms leading to convolution integrals in $L_2(\Phi_0)$ and $L_1(\Phi_1)$ may be ignored. Thus in L_2 only those coefficients having singular parts in their spectra need be considered. Moreover, they need not be considered unless their time-dependent parts, when combined with the $\exp(\pm\tfrac{1}{2}it)$ of Φ_0, lead to a time dependence of the same form as Φ_0. Now the leading term for ϕ is of the form (Havelock 1929)

$$\phi_1 = G(x,z)\cos t + \text{progressive wave train},$$

and while there are many terms in ϕ_2 the only spatially independent term, with a suitable frequency, is that which comes from a pure wave train. However, it can be shown that this field does not contribute to the equation for Φ. Thus ϕ_2 contributes nothing to L_2. The only terms of significance in L_2 are the quadratic terms in ϕ_1 coming from the extended form of (2). For instance, the terms $\tfrac{1}{2}\Phi_{0zzz}\eta^2$ and $\Phi_{0zzz}\eta\zeta$ must be added to the left-hand side of (2*a*), and it is apparent that a term such as η^2 will contain a term, deriving from the wave train only, which must be considered. Now Φ_0 has a smooth continuous spectrum for small wavenumbers for the instability under consideration, and hence the equation for Φ_1 implies that Φ_1 also has a smooth spectrum, and $L_1(\Phi_1)$ will involve only convolutions. Hence Φ_1 need not be calculated.

From such considerations it follows that, for an instability investigation for a mode of the type discussed in § 3, the appropriate modifications of (2) are

$$\Phi_z+\tfrac{1}{2}\eta_1^2\Phi_{zzz}+(\phi_{1zzz}\eta_1)\,\zeta = \zeta_t+\phi_{1x}\zeta_x+\Phi_x\eta_{1x}-\Phi_{zz}\eta_{1x}-\phi_{1zz}\zeta+R_1 \quad (12a)$$

and

$$\Phi_t+\tfrac{1}{2}\eta_1^2\Phi_{tzz}+(\Phi_{1tzz}\eta_1)\,\zeta+(\phi_{1xz}\eta_1)\,\Phi_x+(\phi_{1zz}\eta_1)\,\Phi_z+(\phi_{1z}\eta)\,\Phi_{zz}+\zeta = -\{\Phi_{tz}\eta_1+\phi_{1tz}+\phi_{1x}\Phi_x+\phi_{1z}\Phi_z\}+R_2, \quad (12b)$$

where R_1 and R_2 denote terms which do not affect the stability calculations. Equations (12) have been arranged so that on the left-hand sides only the progressive-wave-train parts of ϕ_1 and η_1 contribute, while on the right-hand sides only convolution integrals with integrands non-singular for small wavenumbers arise. A further simplification results if one observes that in the convolution integrals, to appear eventually, it is only the values of the coefficients at

$k = 0 = k'$ that matter. Thus, on the right-hand sides ϕ_x may be ignored since it will appear as $i(k-k')\hat{\phi}(k-k')$. Similar arguments simplify the calculation of terms like ϕ_{zz}. The above outline should provide a guide as to how the calculations proceed.

In order to obtain the equation corresponding to (4) it is necessary to have the first-order solution for the flow field due to a wavemaker. The results due to Havelock (1929) are not suited to present purposes, for the parasitic modes are calculated in a series of eigenfunctions whereas here they are needed in a Fourier-integral representation. A suitable form may be calculated using a Fourier cosine transform defined by

$$\hat{\phi}(k,z,t) = 2\int_0^\infty \phi(x,z,t)\cos kx\,dx = \int_{-\infty}^\infty \phi_e(x,z,t)\,e^{-ikx}\,dx,$$

where ϕ_e denotes the even extension of ϕ. If one assumes a solution proportional to $\cos t$, as indicated by the boundary condition on the wavemaker, it is easy to show that its Fourier transform is

$$2\cos t\left\{\int_{-\beta}^{z} F(s)\,k^{-1}\sinh k(z-s)\,ds + \cosh k(z+\beta)\,[\cosh k\beta - k\sinh k\beta]^{-1} \times \int_{-\beta}^{0} F(s)\,[\cosh s + k^{-1}\sinh ks]\,ds\right\}.$$

This has poles at $k = \pm l$, where l is the unique positive root of

$$l\tanh l\beta = 1,$$

and the above field neither vanishes at infinity nor satisfies the radiation condition. The necessary correction can be made by adding a suitable multiple of the out-of-phase eigenfunction $\cos lx\sin t\cosh l(z+\beta)$. After some calculation it may be shown that the progressive wave train is given by

$$\phi = A\,\text{sech}\,l\beta\cosh l(z+\beta)\sin(lx-t)$$

and
$$\eta = A\cos(lx-t),$$

where
$$A = 4\cosh l\beta|2l\beta + \sinh 2l\beta|^{-1}\int_{-\beta}^{0} F(s)\cosh l(s+\beta)\,ds,$$

in agreement with the results obtained by Havelock (1929). The continuous spectrum is of concern only at zero wavenumber, and it may be shown that on the surface

$$\hat{\phi}(0,0,t) = 2\int_{-\beta}^{0} F(s)\,ds\cos t = \mu\cos t,$$

$$\hat{\phi}_Z(0,0,t) = \mu\cos t,$$

$$\hat{\phi}_{ZZ}(0,0,t) = 2F(0)\cos t = \nu\cos t,$$

$$\hat{\eta}(0,t) = -\hat{\phi}_t(0,0,t) = \mu\sin t,$$

and moreover all the corresponding functions are smooth at $k = 0$. It is of interest to note that $\hat{\eta}(0,t)$ is related to the total elevation of the surface, and this provides the appropriate generalization of Garrett's mean quantity. Here, too, the properties of ϕ relevant to the instability may be related directly to the performance of the wavemaker, thus maintaining the close connexion with Garrett's calculations.

Furthermore, in evaluating the convolution integrals and the second-order terms on the left-hand side of the equations, it suffices to use the leading-order expressions for Φ, so that one may take

$$\Phi_{0tt} = -\tfrac{1}{4}\hat{\Phi}_0, \quad \Phi_{0z} = \tfrac{1}{4}\hat{\Phi}_0, \quad \bar{\zeta} = -\hat{\Phi}_{0t}, \quad \hat{\Phi}_{0zz} = \alpha^2\hat{\Phi}_0$$

in evaluating the second-order terms. Thus, after some algebra, one is led to the equations

$$\hat{\Phi}_z + \tfrac{1}{4}\epsilon^2 A^2\alpha^2\hat{\Phi} = \zeta_t + (2\pi)^{-1}\epsilon\left\{\nu\cos t\int_{-\infty}^{\infty}\hat{\Phi}_t\,dk' - \alpha^2\mu\sin t\int_{-\infty}^{\infty}\hat{\Phi}\,dk'\right\} + R_3$$

and

$$\hat{\Phi}_t + \zeta + (\tfrac{1}{4}\epsilon^2A^2\alpha^2 - \tfrac{1}{2}\epsilon^2 l^2 A^2)\,\hat{\Phi}_t$$
$$= -(2\pi)^{-1}\epsilon\left\{\tfrac{5}{4}\mu\sin t\int_{-\infty}^{\infty}\hat{\Phi}_t\,dk' + \tfrac{1}{4}\mu\cos t\int_{-\infty}^{\infty}\hat{\Phi}\,dk'\right\} + R_4.$$

Elimination of ζ then yields the equation

$$\hat{\Phi}_{tt} + \{\Omega^2 + \tfrac{1}{16}\epsilon^2 A^2(3\alpha^2+2l^2)\}\,\hat{\Phi}$$
$$= (2\pi)^{-1}\epsilon\left\{(\nu - \tfrac{3}{2}\mu)\cos t\int_{-\infty}^{\infty}\hat{\Phi}_t\,dk' + (\tfrac{9}{16}-\alpha^2)\,\mu\sin t\int_{-\infty}^{\infty}\hat{\Phi}\,dk'\right\}. \quad (13)$$

The same scale changes as in § 3 are introduced and a solution in the form

$$\hat{\Phi} = V(\kappa,\tau)\cos\tfrac{1}{2}t + W(\kappa,\tau)\sin\tfrac{1}{2}t$$

is sought. The usual argument that $\hat{\Phi}$ should be a bounded function of t then yields equations for V and W:

$$V_\tau - \Delta W = -\sigma\int_{-\infty}^{\infty} V\,d\kappa' = X, \quad (13a)$$

$$W_\tau + \Delta V = \sigma\int_{-\infty}^{\infty} W\,d\kappa' = Y, \quad (13b)$$

where
$$\Delta = \lambda + \gamma\kappa^2 + \tfrac{1}{16}A^2(3\alpha^2+2l^2) \quad (13c)$$
and
$$\sigma = \tfrac{1}{4}(2\pi)^{-1}\,[\nu - \mu(\tfrac{3}{8}+2\alpha^2)].$$

The calculation proceeds very much as before. Once again the exponentially growing solutions are associated with positive roots of

$$1 - \sigma^2(p^2\bar{K}_1^2 + \bar{K}_2^2) = 0,$$

and it follows that there is an instability if

$$|\lambda + \tfrac{1}{16}A^2(3\alpha^2+2l^2)| < \{\tfrac{1}{8}\gamma^{-\frac{1}{2}}|\nu - \mu(\tfrac{3}{8}+2\alpha^2)|\}^2.$$

This may be rearranged as

$$|\alpha\tanh\alpha\beta - \tfrac{1}{4} + \tfrac{1}{16}\epsilon^2A^2(3\alpha^2+2l^2)|^{\frac{1}{2}} < 2\alpha(2+8\alpha^2\beta - \tfrac{1}{2}\beta)^{-\frac{1}{2}}\epsilon^2$$
$$\times\left|F(0) - (\tfrac{3}{8}+2\alpha^2)\int_{-\beta}^{0}F(z)\,dz\right|. \quad (14)$$

It may be noted that the inclusion of the second-order terms produces only a small shift in the frequency at which cross-waves are produced, but otherwise has no effect on the instability.

Certain features of the above analysis indicate that it should be possible to facilitate the application of similar arguments in other circumstances. Thus the location of wavenumbers in the neighbourhood of which such instabilities can occur can be obtained solely from the dispersion relation. The effect can only occur at wavenumbers for which the group velocity of the mode describing the instability vanishes. This is in line with the physical argument that, if the group velocity did not vanish, any energy, being fed slowly into this mode, would be radiated away on a much faster time scale and no real growth of the mode would be expected. This simplifies the search for modes which could grow. Further, in deriving what corresponds to the right-hand side of (13), the coefficients, the factor $(2\pi)^{-1}$ apart, may be obtained by a first-order discrete-mode analysis. This follows since the results apply if $\overline{\Phi}$ is simply a δ-function, but may also be verified by examining how the terms arise in the present and Garrett's development. Thus there is an essential discrepancy between formula (14) above and Garrett's equation (3.18). Both results are consistent with Garrett's results for deep water but not otherwise. The present author has been unable to derive Garrett's equation (3.18) and concludes that it is in error.

5. Discussion

There are certain implications of the differences between the present results and Garrett's which are of interest. The first is the result of the smaller order of magnitude of the resonant bandwidth and the rate of growth of the cross-waves. Both here and in Garrett's paper the effect of viscous dissipation has been neglected. The major effect is that the losses in the boundary layers would cause the cross-waves to have a small damping, and in order that the cross-waves occur it is necessary for their growth rate to exceed the decay rate of the cross-wave mode. Thus there is a minimum amplitude of excitation before cross-waves will appear. This implies for the closed channel, which is not too long, that ϵ must exceed a certain critical value, but for the open-ended channel ϵ^2 must exceed a critical value which differs from the first although it is of the same order of magnitude. Thus the threshold amplitude will be significantly greater for the open channel. As for the closed channel, there is a mode of operation of a wavemaker which renders the generation of cross-waves less likely. This is achieved with an operation such that

$$F(0) = (\tfrac{3}{8}+2\alpha^2)\int_{-\beta}^{0} F(z)\,dz, \tag{15}$$

where in this formula it may be taken that α satisfies $\alpha \tanh \alpha\beta = \frac{1}{4}$. Because of the awkward dependence on the parameters, it follows that there would be no simple way of achieving this condition over a range of parameters other than by making both terms vanish. Conventional wavemakers do not operate with both no bulk movement of the fluid and no surface displacement. For a linear wavemaker the above condition reduces to

$$1 = 2\pi\, db^{-1} \tanh \pi d/b\{\tfrac{3}{8} + 2\alpha^2\}.$$

There is a surprising feature of (14) which calls for comment. Suppose $F(z)$ corresponded to the velocity distribution in a travelling wave with the given frequency. Then the wavemaker would generate a pure travelling wave and there would be no non-propagating field, but (14) implies that there are values of α and β for which the cross-wave instability would still exist. Clearly the cross-wave field cannot be said to be due solely to the non-propagating field. But it has been argued that a pure water-wave train will not generate cross-waves. The resolution of this apparent paradox may be illustrated quite simply mathematically. The cross-wave field is to be solved subject to a boundary condition on the wavemaker which has been taken to be $\Phi_x = 0$. This has been achieved by solving for an even Fourier transform $\hat{\Phi}$ which implies using an even continuation of ϕ. However, the travelling-wave part of ϕ contains a term proportional to $\sin lx \cos t$, and its even continuation has a Fourier transform which has poles at $\pm l$ and is continuous elsewhere; this is in marked contrast with the δ-function behaviour of the Fourier transform of a pure travelling wave. In the presence of the wavemaker, the wave train contributes significantly to the cross-wave instability. Physically, the role of the wavemaker may be described as a sink for momentum so that energy can be transferred with momentum conservation. This has the clear implication that the amplitude of the wave train produced will be affected by the generation of cross-waves rather more directly than by changes in the non-propagating field and hence the phases.

Although the present analysis is concerned only with the initiation of cross-waves, it appears reasonable to assume that the manner in which the growing disturbance varies along the channel will be at least partially reflected in the established cross-wave field. If this is so, the present analysis offers an explanation for the decay of the cross-waves down the channel, which is not associated with a dissipative mechanism. The disturbance, which is growing exponentially in time, has a spatial structure the details of which can be determined from the instability calculations. As it is not clear that this spatial structure will be reflected in the cross-wave field when fully established, the detailed calculations will not be presented but the general form may be seen from (13a) and (13b). When the instability occurs X and Y have $\exp(s\tau)$ as a factor and so both V and W have the form

$$(A_1 s + B_1 \Delta)/(s^2 + \Delta^2),$$

where A_1 and B_1 are appropriate constants. The variation with x of the developing cross-wave can thus be obtained by an appropriate Fourier inversion using also the fact that s is independent of κ and Δ involves κ, as is implied by (13c). Clearly the length scale of the cross-wave field is of $O(\epsilon^{-1})$ and, furthermore, the decay is exponential for large values of ϵx. It may be noted that this decay along the channel will occur even in the idealized limit of vanishing dissipation. The result also implies that if this effect, rather than dissipation, controls the variation of cross-wave pattern along the channel then the stronger the wavemaker action the less the extent of the cross-waves.

One of the difficulties associated with the understanding of cross-waves is the mechanism whereby energy supplied by the wavemaker is subsequently to be found in the cross-wave mode down the channel, since a pure cross-wave mode does not propagate energy along the channel. The form of the calculations suggests that the energy transfer could take place in the long-wave part of the spectrum and thence into the cross-waves. This is not in conflict with Garrett's argument that the energy balance for the whole system requires that the energy in the cross-waves be supplied by the wavemaker. However, in the absence of a method of calculating the field near the wavemaker when cross-waves are established, there appears to be no way to settle the manner in which the wavemaker works against the stresses associated with the presence of cross-waves.

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